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Mathematical treatise: [kg*m^2/s] and [radians/s] conversion

Converting angular momentum [kg*m^2/s] to angular velocity [radians/s]

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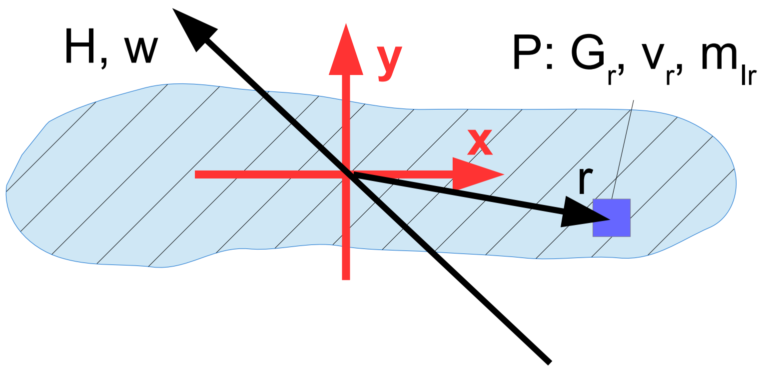

$\vec{H}$ Angular momentum
$\vec{w}$ Angular velocity
$\vec{r}$ Offset
$\vec{G}_r$ Particle linear momentum
$\vec{v}_r$ Particle linear velocity
$\vec{m}_{Ir}$ Particle mass vector
I Moment of Inertia Tensor

$$I = [\vec{I}_x, \vec{I}_y, \vec{I}_z] = \begin{bmatrix} \vec{I}_{x.x} & \vec{I}_{y.x} & \vec{I}_{z.x} \\ \vec{I}_{x.y} & \vec{I}_{y.y} & \vec{I}_{z.y} \\ \vec{I}_{x.z} & \vec{I}_{y.z} & \vec{I}_{z.z} \end{bmatrix}$$

At first we thought $|H| = Iw$ meant $\vec{H} = I\vec{w} \rightarrow \vec{w} = I^{-1}\vec{H}$. Which turns out being wrong when we ran it in our impulse-momentum-based physics engine. So it was decided to crack down how to reliable convert angular momentum $\vec{H}$ $kg*m^2/s$ to angular velocity $\vec{w}$ $radian/s$.

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There is a possibility that the moment of Inertia tensor I is not aligned with the base axis → not diagonolized. Which is the preferred case. This will be solved by making sure that I is diagonolized.

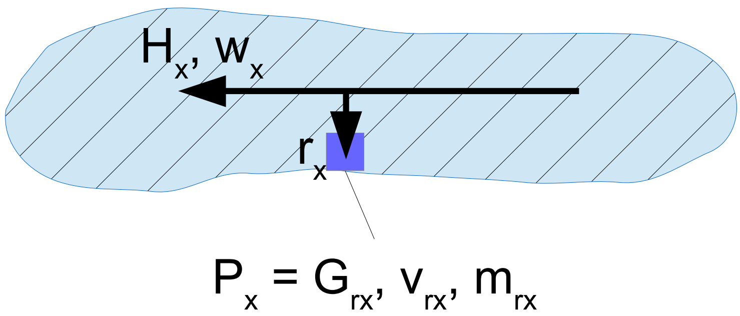
We could say that the I tensor got 2 properties; magnitudescalars I_S and rotation I_R.

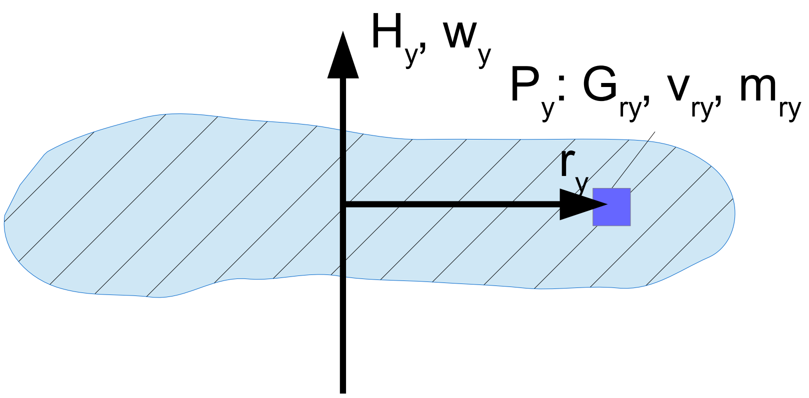
$$I = I_R I_S = \left[\vec{I}_{Rx} \quad \vec{I}_{Ry} \quad \vec{I}_{Rz}\right]\begin{bmatrix} I_{Sxx} & 0 & 0 \\ 0 & I_{Syy} & 0 \\ 0 & 0 & I_{Szz} \end{bmatrix} = \begin{bmatrix} I_{Rxx} & I_{Ryx} & I_{Rzx} \\ I_{Rxy} & I_{Ryy} & I_{Rzy} \\ I_{Rxz} & I_{Ryz} & I_{Rzz} \end{bmatrix}\begin{bmatrix} I_{Sxx} & 0 & 0 \\ 0 & I_{Syy} & 0 \\ 0 & 0 & I_{Szz} \end{bmatrix}$$

$$I = \begin{bmatrix} I_{Sxx} I_{Rxx} & I_{Syy} I_{Ryx} & I_{Szz} I_{Rzx} \\ I_{Sxx} I_{Rxy} & I_{Syy} I_{Ryy} & I_{Szz} I_{Rzy} \\ I_{Sxx} I_{Rxz} & I_{Syy} I_{Ryz} & I_{Szz} I_{Rzz} \end{bmatrix} = \left[I_{Sxx}\vec{I}_{Rx} \quad I_{Syy}\vec{I}_{Ry} \quad I_{Szz}\vec{I}_{Rz}\right]$$

From this point on is $i \in \{x, y, z\}$ to avoid redundancy in this document.
To ensure that the moment of inertia tensor I is diagonolized, we'll transform everything into the local space of $I_R \rightarrow I'_R$ = identitymatrix $[\hat{e}_x, \quad \hat{e}_y, \quad \hat{e}_z]$.

$$I'_R = [\hat{e}_x, \quad \hat{e}_y, \quad \hat{e}_z] = I_R^{-1} * I_R$$
$$\vec{H}' = I_R^{-1}\vec{H}$$
$$\vec{w}' = I_R^{-1}\vec{w}$$
$$\vec{r}' = I_R^{-1}\vec{r}$$
$$\vec{G}'_r = I_R^{-1}\vec{G}_r$$
$$\vec{v}'_r = I_R^{-1}\vec{v}_r$$
$$m'_{ri}$$





$$\vec{H}'_i = (\vec{H}' \cdot \hat{e}_i)\hat{e}_i$$
$$\vec{r}'_i = \vec{r}' - (\vec{r}' \cdot \hat{e}_i)\hat{e}_i$$
$$\vec{w}'_i = (\vec{w}' \cdot \hat{e}_i)\hat{e}_i = \vec{r}'_i \times \vec{v}'_{ri}$$
$$\vec{G}'_{ri} = m'_{ri}\vec{v}'_{ri} \rightarrow$$
$$\rightarrow \vec{v}'_{ri} = \vec{G}'_{ri}\frac{1}{m'_{ri}}$$
$$\vec{H}'_i = \vec{r}'_i \times \vec{G}'_r \rightarrow$$
$$\rightarrow \vec{G}'_{ri} = (\vec{H}'_i \times \vec{r}'_i) * \frac{1}{|\vec{r}'_i|^2}$$

$$I_{pi} = m'_{ri}|\vec{r}'_i|^2 \rightarrow m'_{ri} = \frac{I_{pi}}{|\vec{r}'_i|^2}, \quad \{I_{pi} = I_{Sii}\} \rightarrow m'_{ri} = \frac{I_{Sii}}{|\vec{r}'_i|^2}$$

$$\vec{w}'_i = \vec{r}'_i \times \vec{v}'_{ri} = \vec{r}'_i \times \left(\vec{G}'_{ri}\frac{1}{m'_{ri}}\right) = \vec{r}'_i \times \left(\vec{G}'_{ri}\frac{1}{\left(\frac{I_{Sii}}{|\vec{r}'_i|^2}\right)}\right) = \vec{r}'_i \times \left(\vec{G}'_{ri}\frac{|\vec{r}'_i|^2}{I_{Sii}}\right)$$

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$$\vec{w}'_i = \vec{r}'_i \times (((\vec{H}'_i \times \vec{r}'_i)\frac{1}{|\vec{r}'_i|^2})\frac{|\vec{r}'_i|^2}{I_{Sii}}) = (\vec{r}'_i \times \vec{H}'_i \times \vec{r}'_i)\frac{1}{I_{Sii}}$$

$$\{\vec{a} \times \vec{b} \times \vec{a} = |\vec{a}|^2 \vec{b}\} \rightarrow \vec{w}'_i = |\vec{r}'_i|^2 \vec{H}'_i \frac{1}{I_{Sii}} = \frac{|\vec{r}'_i|^2}{I_{Sii}} * \vec{H}'_i$$

To wrap it up, we sum the 3 local axis cases together and then transform into our targeted angular velocity $\vec{w}$.

$$\vec{w}' = \sum^{i \in \{x,y,z\}} \vec{w}'_i = \vec{w}'_x + \vec{w}'_y + \vec{w}'_z = \frac{|\vec{r}'_x|^2}{I_{Sxx}}\vec{H}'_x + \frac{|\vec{r}'_y|^2}{I_{Syy}}\vec{H}'_y + \frac{|\vec{r}'_z|^2}{I_{Szz}}\vec{H}'_z$$

$$\vec{w}' = \frac{|\vec{r}' - (\vec{r}' \cdot \hat{e}_x)\hat{e}_x|^2}{I_{Sxx}}((\vec{H}' \cdot \hat{e}_x)\hat{e}_x) + \frac{|\vec{r}' - (\vec{r}' \cdot \hat{e}_y)\hat{e}_y|^2}{I_{Syy}}((\vec{H}' \cdot \hat{e}_y)\hat{e}_y) + \frac{|\vec{r}' - (\vec{r}' \cdot \hat{e}_z)\hat{e}_z|^2}{I_{Szz}}((\vec{H}' \cdot \hat{e}_z)\hat{e}_z)$$

$$\vec{w}' = \left(\frac{(r'_y)^2 + (r'_z)^2}{I_{Sxx}} H'_x ,\quad \frac{(r'_x)^2 + (r'_z)^2}{I_{Syy}} H'_y ,\quad \frac{(r'_x)^2 + (r'_y)^2}{I_{Szz}} H'_z\right)$$

$$\vec{w} = I_R \vec{w}' \quad [m/s]$$

Hold on! $\vec{r}$ should not be a factor when it comes to a direct $H[kg\, m^2/s] \rightarrow w[\text{radians}/s]$ conversion. And you would be correct to state that. The thing is that $\text{radians}/s$ is directly equivalent with m/s if and only if $|\vec{r}| = 1$.

Set $\vec{r}' = (\sqrt{3} ,\ \sqrt{3} ,\ \sqrt{3})$

$$|\vec{w}'| = \left|\left(\frac{3+3}{I_{Sxx}} H'_x ,\quad \frac{3+3}{I_{Syy}} H'_y ,\quad \frac{3+3}{I_{Szz}} H'_z\right)\right|$$

$$|\vec{w}'| = \left|\left(\frac{6}{I_{Sxx}} H'_x ,\quad \frac{6}{I_{Syy}} H'_y ,\quad \frac{6}{I_{Szz}} H'_z\right)\right|$$

$$|\vec{w}| = |\vec{w}'| = |6 * I_S^{-1} \vec{H}'| = |6 * I_S^{-1} I_R^{-1} \vec{H}| = |6 * I^{-1} \vec{H}|$$

$$\vec{w} = |\vec{w}|\hat{w} = |\vec{w}|\hat{H} = \frac{|\vec{w}|}{|\vec{H}|}\vec{H} = \frac{|6 * I_S^{-1} I_R^{-1} \vec{H}|}{|\vec{H}|}\vec{H} = \frac{|6 * I^{-1} \vec{H}|}{|\vec{H}|}\vec{H} \quad [\text{radians}/s]$$

And there we have it! A direct conversion from $kg\, m^2/s$ to $\text{radians}/s$.
Howether, before we celebrate. We'll need a way to directly convert $\text{radians}/s$ to $kg\, m^2/s$.

$$\vec{w}'_i = \vec{r}'_i \times \vec{v}'_{ri} \rightarrow \vec{v}'_{ri} = (\vec{w}'_i \times \vec{r}'_i)\frac{1}{|\vec{r}'_i|^2} = \vec{G}'_{ri}\frac{1}{m'_{ri}} = ((\vec{H}'_i \times \vec{r}'_i)\frac{1}{|\vec{r}'_i|^2})\frac{1}{\left(\frac{I_{Sii}}{|\vec{r}'_i|^2}\right)} = ((\vec{H}'_i \times \vec{r}'_i)\frac{1}{|\vec{r}'_i|^2})\frac{|\vec{r}'_i|^2}{I_{Sii}}$$

$$(\vec{w}'_i \times \vec{r}'_i)\frac{1}{|\vec{r}'_i|^2} = (\vec{H}'_i \times \vec{r}'_i)\frac{1}{I_{Sii}} \rightarrow (\vec{w}'_i \times \vec{r}'_i)\frac{I_{Sii}}{|\vec{r}'_i|^2} = \vec{H}'_i \times \vec{r}'_i \rightarrow \vec{H}'_i = (\vec{r}'_i \times ((\vec{w}'_i \times \vec{r}'_i)\frac{I_{Sii}}{|\vec{r}'_i|^2}))\frac{1}{|\vec{r}'_i|^2}$$

$$\vec{H}'_i = (\vec{r}'_i \times \vec{w}'_i \times \vec{r}'_i)\frac{I_{Sii}}{|\vec{r}'_i|^4} = |\vec{r}'_i|^2 \vec{w}'_i \frac{I_{Sii}}{|\vec{r}'_i|^4} = \frac{I_{Sii}}{|\vec{r}'_i|^2}\vec{w}'_i$$

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$$\vec{H}' = \sum^{i\in\{x,y,z\}} \vec{H}'_i = \vec{H}'_x+\vec{H}'_y+\vec{H}'_z = \frac{I_{Sxx}}{|\vec{r}'_x|^2}\vec{w}'_x+\frac{I_{Syy}}{|\vec{r}'_y|^2}\vec{w}'_y+\frac{I_{Szz}}{|\vec{r}'_z|^2}\vec{w}'_z$$

$$\vec{H}' = \frac{I_{Sxx}}{|\vec{r}'-(\vec{r}'\cdot\hat{e}_x)\hat{e}_x|^2}((\vec{H}'\cdot\hat{e}_x)\hat{e}_x)+\frac{I_{Syy}}{|\vec{r}'-(\vec{r}'\cdot\hat{e}_Y)\hat{e}_Y|^2}((\vec{H}'\cdot\hat{e}_y)\hat{e}_y)+\frac{I_{Szz}}{|\vec{r}'-(\vec{r}'\cdot\hat{e}_z)\hat{e}_z|^2}((\vec{H}'\cdot\hat{e}_z)\hat{e}_z)$$

$$\vec{H}' = \left(\frac{I_{Sxx}}{(r'_y)^2+(r'_z)^2}w'_x\ ,\quad \frac{I_{Syy}}{(r'_x)^2+(r'_z)^2}w'_y\ ,\quad \frac{I_{Szz}}{(r'_x)^2+(r'_y)^2}w'_z\right)$$

$\vec{H} = I_R*\vec{H}'\ [kg\,m^2/s]$

Set $\vec{r}' = (\sqrt{3}\ ,\quad \sqrt{3}\ ,\quad \sqrt{3})$

$$|\vec{H}'| = \left|\left(\frac{I_{Sxx}}{3+3}w'_x\ ,\quad \frac{I_{Syy}}{3+3}w'_y\ ,\quad \frac{I_{Szz}}{3+3}w'_z\right)\right| = \left|\left(\frac{I_{Sxx}}{6}w'_x\ ,\quad \frac{I_{Syy}}{6}w'_y\ ,\quad \frac{I_{Szz}}{6}w'_z\right)\right|$$

$$|\vec{H}| = |\vec{H}'| = \left|\frac{1}{6}I_S\vec{w}'\right| = \left|\frac{1}{6}I_S I_R^{-1}\vec{w}\right|$$

$$\vec{H} = |\vec{H}|\hat{H} = |\vec{H}|\hat{w} = \frac{|\vec{H}|}{|\vec{w}|}\vec{w} = \frac{\left|\frac{1}{6}I_S I_R^{-1}\vec{w}\right|}{|\vec{w}|}\vec{w}\ [kg\,m^2/s]$$

Summary

$$\vec{w} = \frac{|6*I_S^{-1}I_R^{-1}\vec{H}|}{|\vec{H}|}\vec{H} = \frac{|6*I^{-1}\vec{H}|}{|\vec{H}|}\vec{H}\ [\text{radians}/s]$$

$$\vec{H} = \frac{\left|\frac{1}{6}I_S I_R^{-1}\vec{w}\right|}{|\vec{w}|}\vec{w}\ [kg\,m^2/s]$$