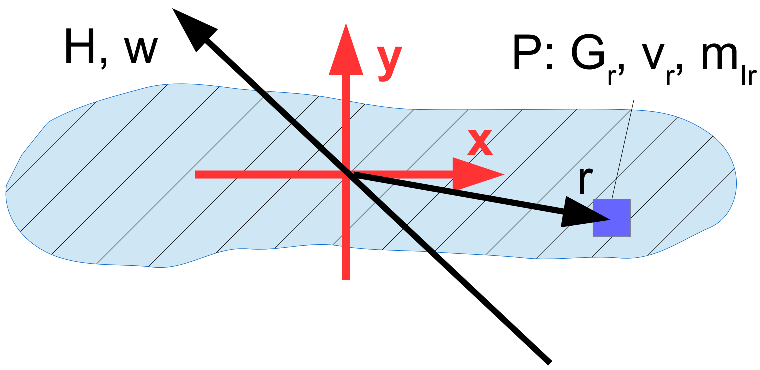
Dan Andersson, daaf11@student.bth.se 2014
Linda Andersson, liaj11@student.bth.se 2014

Converting angular momentum [kg*m^2/s] to angular velocity [radians/s]

by Dan Andersson

assisted by Linda Andersson



$\vec{H}$ Angular momentum

$\vec{w}$ Angular velocity

$\vec{r}$ Offset

$\vec{G}_r$ Particle linear momentum

$\vec{v}_r$ Particle linear velocity

$\vec{m}_{Ir}$ Particle mass vector

I Moment of Inertia Tensor

$$I = \left[\vec{I}_x , \quad \vec{I}_y , \quad \vec{I}_z\right] = \begin{bmatrix} \vec{I}_{x.x} & \vec{I}_{y.x} & \vec{I}_{z.x} \\ \vec{I}_{x.y} & \vec{I}_{y.y} & \vec{I}_{z.y} \\ \vec{I}_{x.z} & \vec{I}_{y.z} & \vec{I}_{z.z} \end{bmatrix}$$

At first we thought $|H| = Iw$ meant $\vec{H} = I\vec{w} \rightarrow \vec{w} = I^{-1}\vec{H}$. Which turns out being wrong when we ran it in our impulse-momentum-based physics engine. So it was decided to crack down how to reliable convert angular momentum $\vec{H}$ $kg*m^2/s$ to angular velocity $\vec{w}$ $radian/s$.

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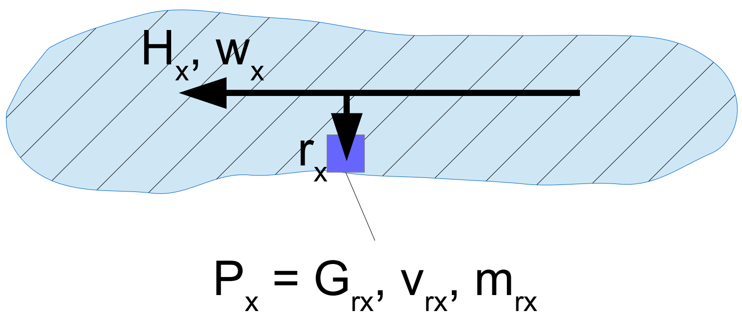


$$\vec{H}_x = (\vec{H}\cdot\hat{x})*\hat{x}$$
$$\vec{r}_x = \vec{r}-(\vec{r}\cdot\hat{x})*\hat{x}$$
$$\vec{w}_x = (\vec{w}\cdot\hat{x})*\hat{x} = \vec{r}_x\times\vec{v}_{rx}$$

$$\vec{G}_{rx} = m_{Irx}*\vec{v}_{rx} \rightarrow$$
$$\rightarrow \vec{v}_{rx} = \vec{G}_{rx}*\frac{1}{m_{Irx}}$$

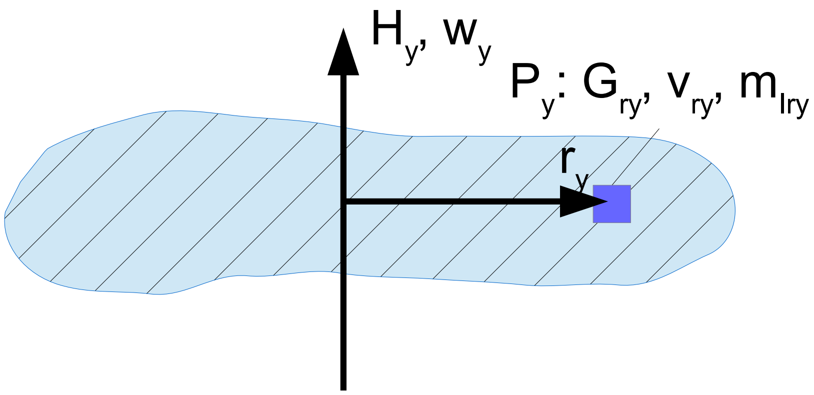
$$\vec{H}_x = \vec{r}_x\times\vec{G} \rightarrow$$
$$\rightarrow \vec{G}_{ry} = (\vec{H}_x\times\vec{r}_x)*\frac{1}{|\vec{r}_x|^2}$$

$$I_{px} = m_{Irx}*|\vec{r}_x|^2 \rightarrow m_{Irx} = \frac{I_{px}}{|\vec{r}_x|^2} \quad , \quad \{I_{px} = \vec{I}_{x.x}\} \rightarrow m_{Irx} = \frac{\vec{I}_{x.x}}{|\vec{r}_x|^2}$$

$$\vec{w}_x = \vec{r}_x\times\vec{v}_{rx} = \vec{r}_x\times(\vec{G}_{rx}*\frac{1}{m_{Irx}}) = \vec{r}_x\times(\vec{G}_{rx}*\frac{1}{(\frac{\vec{I}_{x.x}}{|\vec{r}_x|^2})}) = \vec{r}_x\times(\vec{G}_{rx}*\frac{|\vec{r}_x|^2}{\vec{I}_{x.x}})$$

$$\vec{w}_x = \vec{r}_x\times(((\vec{H}_x\times\vec{r}_x)*\frac{1}{|\vec{r}_x|^2})*\frac{|\vec{r}_x|^2}{\vec{I}_{x.x}}) = \vec{r}_x\times\vec{H}_x\times\vec{r}_x*\frac{1}{\vec{I}_{x.x}}$$

$$\{\vec{a}\times\vec{b}\times\vec{a}=|\vec{a}|^2*\vec{b}\} \rightarrow \vec{w}_x = |\vec{r}_x|^2*\vec{H}_x*\frac{1}{\vec{I}_{x.x}} = \frac{|\vec{r}_x|^2}{\vec{I}_{x.x}}*\vec{H}_x$$



$\vec{w}_y$ and $\vec{w}_z$ is deduced in the same way as $\vec{w}_x$.

$$\vec{w}_x = \frac{|\vec{r}_x|^2}{\vec{I}_{x.x}}*\vec{H}_x$$
$$\vec{w}_y = \frac{|\vec{r}_y|^2}{\vec{I}_{y.y}}*\vec{H}_y$$
$$\vec{w}_z = \frac{|\vec{r}_z|^2}{\vec{I}_{z.z}}*\vec{H}_z$$

The angular velocity $\vec{w}$ is the sum of $\vec{w}_x$, $\vec{w}_y$ and $\vec{w}_z$.

$$\vec{w} = \vec{w}_x+\vec{w}_y+\vec{w}_z$$
$$\vec{w} = \frac{|\vec{r}_x|^2}{\vec{I}_{x.x}}*\vec{H}_x+\frac{|\vec{r}_y|^2}{\vec{I}_{y.x}}*\vec{H}_y+\frac{|\vec{r}_z|^2}{\vec{I}_{z.z}}*\vec{H}_z$$

$$\vec{w} = \frac{|\vec{r}-(\vec{r}\cdot\hat{x})*\hat{x}|^2}{\vec{I}_{x.x}}*(\vec{H}\cdot\hat{x})*\hat{x}+\frac{|\vec{r}-(\vec{r}\cdot\hat{y})*\hat{y}|^2}{\vec{I}_{y.y}}*(\vec{H}\cdot\hat{y})*\hat{y}+\frac{|\vec{r}-(\vec{r}\cdot\hat{z})*\hat{z}|^2}{\vec{I}_{z.z}}*(\vec{H}\cdot\hat{z})*\hat{z}$$

$$\vec{w} = \left(\frac{(\vec{r}_2)^2+(\vec{r}_3)^2}{\vec{I}_{x.x}}*\vec{H}_1 , \quad \frac{(\vec{r}_1)^2+(\vec{r}_3)^2}{\vec{I}_{y.y}}*\vec{H}_2 , \quad \frac{(\vec{r}_1)^2+(\vec{r}_2)^2}{\vec{I}_{z.z}}*\vec{H}_3\right)$$

Is $m^2/s \rightarrow radian/s$? Is $\vec{w} = \vec{r}\times\vec{v}$ correct to begin with? And do this relate to $|H| = Iw$?

Dan Andersson, daaf11@student.bth.se 2014
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Version 2

There is a possibility that the moment of Inertia tensor I is not aligned with the base axis → not diagonolized. Which is the preferred case. This will be solved by making sure that I is diagonolized.

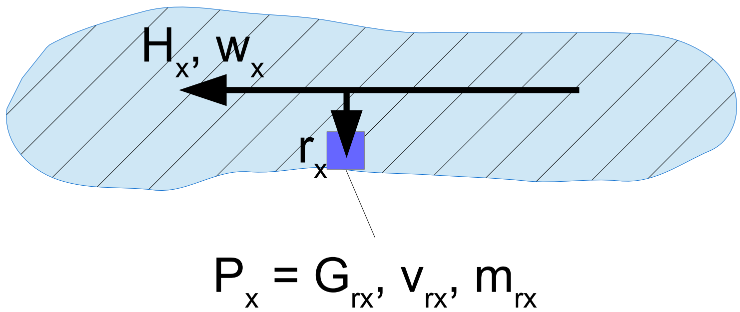
We could say that the I tensor got 2 properties; magnitudescalars I_S and rotation I_R.

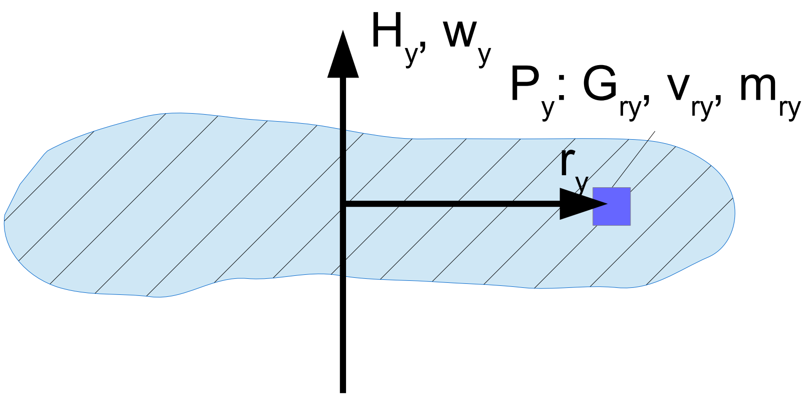
$$I = I_R I_S = \begin{bmatrix} \vec{I}_{Rx} & \vec{I}_{Ry} & \vec{I}_{Rz} \end{bmatrix} \begin{bmatrix} I_{Sxx} & 0 & 0 \\ 0 & I_{Syy} & 0 \\ 0 & 0 & I_{Szz} \end{bmatrix} = \begin{bmatrix} I_{Rxx} & I_{Ryx} & I_{Rzx} \\ I_{Rxy} & I_{Ryy} & I_{Rzy} \\ I_{Rxz} & I_{Ryz} & I_{Rzz} \end{bmatrix} \begin{bmatrix} I_{Sxx} & 0 & 0 \\ 0 & I_{Syy} & 0 \\ 0 & 0 & I_{Szz} \end{bmatrix}$$

$$I = \begin{bmatrix} I_{Sxx} I_{Rxx} & I_{Syy} I_{Ryx} & I_{Szz} I_{Rzx} \\ I_{Sxx} I_{Rxy} & I_{Syy} I_{Ryy} & I_{Szz} I_{Rzy} \\ I_{Sxx} I_{Rxz} & I_{Syy} I_{Ryz} & I_{Szz} I_{Rzz} \end{bmatrix} = \begin{bmatrix} I_{Sxx} \vec{I}_{Rx} & I_{Syy} \vec{I}_{Ry} & I_{Szz} \vec{I}_{Rz} \end{bmatrix}$$

From this point on is $i \in \{x, y, z\}$ to avoid redundancy in this document.
To ensure that the moment of inertia tensor I is diagonolized, we'll transform everything into the local space of $I_R \rightarrow I'_R$ = identitymatrix $[\hat{e}_x, \ \hat{e}_y, \ \hat{e}_z]$.

$$I'_R = [\hat{e}_x, \ \hat{e}_y, \ \hat{e}_z] = I_R^{-1} * I_R$$
$$\vec{H}' = I_R^{-1} \vec{H}$$
$$\vec{w}' = I_R^{-1} \vec{w}$$
$$\vec{r}' = I_R^{-1} \vec{r}$$
$$\vec{G}'_r = I_R^{-1} \vec{G}_r$$
$$\vec{v}'_r = I_R^{-1} \vec{v}_r$$
$$m'_{ri}$$





$$\vec{H}'_i = (\vec{H}' \cdot \hat{e}_i)\hat{e}_i$$
$$\vec{r}'_i = \vec{r}' - (\vec{r}' \cdot \hat{e}_i)\hat{e}_i$$
$$\vec{w}'_i = (\vec{w}' \cdot \hat{e}_i)\hat{e}_i = \vec{r}'_i \times \vec{v}'_{ri}$$
$$\vec{G}'_{ri} = m'_{ri} \vec{v}'_{ri} \rightarrow$$
$$\rightarrow \vec{v}'_{ri} = \vec{G}'_{ri} \frac{1}{m'_{ri}}$$
$$\vec{H}'_i = \vec{r}'_i \times \vec{G}'_r \rightarrow$$
$$\rightarrow \vec{G}'_{ri} = (\vec{H}'_i \times \vec{r}'_i) * \frac{1}{|\vec{r}'_i|^2}$$

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$$I_{pi} = m'_{ri}|\vec{r}'_i|^2 \rightarrow m'_{ri} = \frac{I_{pi}}{|\vec{r}'_i|^2} , \quad \{I_{pi} = I_{Sii}\} \rightarrow m'_{ri} = \frac{I_{Sii}}{|\vec{r}'_i|^2}$$

$$\vec{w}'_i = \vec{r}'_i \times \vec{v}'_{ri} = \vec{r}'_i \times (\vec{G}'_{ri}\frac{1}{m'_{ri}}) = \vec{r}'_i \times (\vec{G}'_{ri}\frac{1}{\left(\frac{I_{Sii}}{|\vec{r}'_i|^2}\right)}) = \vec{r}'_i \times (\vec{G}'_{ri}\frac{|\vec{r}'_i|^2}{I_{Sii}})$$

$$\vec{w}'_i = \vec{r}'_i \times (((\vec{H}'_i \times \vec{r}'_i)\frac{1}{|\vec{r}'_i|^2})\frac{|\vec{r}'_i|^2}{I_{Sii}}) = (\vec{r}'_i \times \vec{H}'_i \times \vec{r}'_i)\frac{1}{I_{Sii}}$$

$$\{\vec{a}\times\vec{b}\times\vec{a} = |\vec{a}|^2\vec{b}\} \rightarrow \vec{w}'_i = |\vec{r}'_i|^2 \vec{H}'_i \frac{1}{I_{Sii}} = \frac{|\vec{r}'_i|^2}{I_{Sii}} * \vec{H}'_i$$

To wrap it up, we sum the 3 local axis cases together and then transform into our targeted angular velocity $\vec{w}$.

$$\vec{w}' = \sum^{i\in\{x,y,z\}} \vec{w}'_i = \vec{w}'_x + \vec{w}'_y + \vec{w}'_z = \frac{|\vec{r}'_x|^2}{I_{Sxx}}\vec{H}'_x + \frac{|\vec{r}'_y|^2}{I_{Syy}}\vec{H}'_y + \frac{|\vec{r}'_z|^2}{I_{Szz}}\vec{H}'_z$$

$$\vec{w}' = \frac{|\vec{r}' - (\vec{r}'\cdot\hat{e}_x)\hat{e}_x|^2}{I_{Sxx}}((\vec{H}'\cdot\hat{e}_x)\hat{e}_x) + \frac{|\vec{r}' - (\vec{r}'\cdot\hat{e}_y)\hat{e}_y|^2}{I_{Syy}}((\vec{H}'\cdot\hat{e}_y)\hat{e}_y) + \frac{|\vec{r}' - (\vec{r}'\cdot\hat{e}_z)\hat{e}_z|^2}{I_{Szz}}((\vec{H}'\cdot\hat{e}_z)\hat{e}_z)$$

$$\vec{w}' = \left(\frac{(r'_y)^2 + (r'_z)^2}{I_{Sxx}} H'_x , \quad \frac{(r'_x)^2 + (r'_z)^2}{I_{Syy}} H'_y , \quad \frac{(r'_x)^2 + (r'_y)^2}{I_{Szz}} H'_z \right) \; [m/s]$$

$$\vec{w} = I_R \vec{w}' \; [m/s]$$

Hold on! $\vec{r}$ should not be a factor when it comes to a direct $H[kg\,m^2/s] \rightarrow w[\text{radians}/s]$ conversion. And you would be correct to state that. The thing is that $\text{radians}/s$ is directly equivalent with m/s if and only if $|\vec{r}'_i| = 1$.

$$\vec{w} = I_R\left(\frac{1}{I_{Sxx}} H'_x , \quad \frac{1}{I_{Syy}} H'_y , \quad \frac{1}{I_{Szz}} H'_z \right) \; [\text{radians}/s]$$

$$\vec{w} = I_R I_S^{-1} \vec{H}' = I_R I_S^{-1} I_R^{-1} \vec{H} = I_R I^{-1} \vec{H} \; [\text{radians}/s]$$

$$\vec{H} = I I_R^{-1} \vec{w} \; [kg\,m^2/s]$$

And there we have it! Direct conversion between $kg\,m^2/s$ and $\text{radians}/s$.